Characteristics of Main Sequence Stars

Main-sequence stars obey several relations (which are mostly predictable from homology).

• Main sequence stars obey a mass-luminosity relation, with $\mathcal{L} \propto \mathcal{M}^{\eta}$. The slope η changes slightly over the range of masses; between 1 and $10\mathcal{M}_{\odot}$, $\eta \approx 3.88$. The relation flattens out at higher masses, due to the contribution of radiation pressure in the central core. (This helps support the star, and decreases the central temperature slightly.) The relation also flattens significantly at the very faint end of the luminosity function. This is due to the increasing important of convection for stellar structure.

• Main sequence stars obey a mass-radius relation. However, the relation displays a significant break around $1\mathcal{M}_{\odot}$; $R \propto \mathcal{M}^{\xi}$, with $\xi \approx 0.57$ for $\mathcal{M} > 1\mathcal{M}_{\odot}$, and $\xi \approx 0.8$ for $\mathcal{M} < 1\mathcal{M}_{\odot}$. This division marks the onset of a convective envelope. Convection tends to increase the flow of energy out of the star, which causes the star to contract slightly. As a result, stars with convective envelopes lie below the mass-radius relation for non-convective stars. (This contraction also increases the central temperature (via the virial theorum) and also moves the star above the nominal mass-luminosity relation.

• Near the surface of a convective star, radiation often plays a role in the energy transport, so that the true value of the adiabatic gradient is $\nabla_{\text{rad}} > \nabla > \nabla_{\text{ad}}$. Solving for ∇ is tricky, as it involves a number of (poor) assumptions, and the adoption of a free parameter called the "mixing length", which is the distance a blob of convective material moves before dissolving into its surroundings. (Actually, the mixing length is usually quoted as the ratio of this distance to the local pressure scale height (i.e., the

length it takes the pressure to decline by $1/e$), so that

$$\alpha = l_m/\lambda_P \quad \text{where} \quad \lambda_P = -\frac{dr}{d\ln P}$$

Changing the mixing length changes ∇ which changes the pressure gradient, which changes the size (and therefore the temperature) of the star. The larger the surface convective zone (or the smaller the pressure gradient), the larger the change in radius.

• The mixing length is essentially a free parameter in the models for stars with convective envelopes. For main sequence stars, uncertainties in the mixing scale length do affect the computed stellar radius (and effective temperature), but only slightly. A factor of two increase in α translates into a change of only $+0.03$ in $\log T_{\text{eff}}$. The star's luminosity, of course, is not affected. First generation stellar models (from the 1960's) adopted $\alpha \sim 1$, but those models produced stars that were too cool. A value of 1.5 or 1.6 does a better job at fitting the observations.

• The depth of the convective envelope (in terms of $\mathcal{M}_{\text{env}}/\mathcal{M}_T$) increases with decreasing mass. Stars with $\mathcal{M} \sim 1\mathcal{M}_\odot$ have extremely thin convective envelopes, while stars with $\mathcal{M} \lesssim 0.3\mathcal{M}_\odot$ are entirely convective. Nuclear burning ceases around $\mathcal{M} \sim 0.08\mathcal{M}_\odot$.

• The interiors of stars are extremely hot ($T > 10^6$ K). The fall-off to surface temperatures ($T \sim 10^4$ K) takes place in a very thin region near the surface.

• The region of nuclear energy generation is restricted to a very small mass range near the center of the star. The rapid fall-off of ϵ_n with radius reflects the extreme sensitivity of energy generation to temperature.

• Stars with masses below $\sim 1\mathcal{M}_\odot$ generate most of their energy via the proton-proton chain. Stars with more mass than this create most of their energy via the CNO cycle. This changeover causes a shift in the homology relations for the stellar interior.

• CNO burning exhibits an extreme temperature dependence. Consequently, those stars that are dominated by CNO fusion have very large values of $\mathcal{L}/4\pi r^2$ in the core. This results in a large value of $\nabla_{\rm rad}$, and convective instability. In this region, convective energy transport is extremely efficient, and $\nabla \approx \nabla_{\rm ad}$.

• Because of the extreme temperature sensitivity of CNO burning, nuclear reactions in high mass stars are generally confined to a very small region, much smaller than the size of the convective core.

• As the stellar mass increases, so does the size of the convective core (due again to the large increase in ϵ with temperature). Supermassive stars with $\mathcal{M} \sim 100\mathcal{M}_\odot$ would be entirely convective.

• A star's position on the ZAMS depends on both its mass and its initial helium abundance. Stars with higher initial helium abundances have higher luminosities and effective temperatures. This is predicted by homology; the higher mean molecular weight translates into lower core pressures. Helium rich stars therefore are more condensed, which (through the virial theorum) mean they have higher core temperatures and larger nuclear reaction rates.

• Changes in metallicity shift the location of the ZAMS; the metal-poor main sequence is blueward of the solar metallicity main sequence. This is primarily due to the reduced amount of bound-free absorption throughout the star (which only comes from metals). The smaller opacity allows the energy to escape more easily, and

thus increases the luminosity via (3.1.4), *i.e.,*

$$\mathcal{L} = -\frac{16\pi acr^2T^3}{3\kappa\rho}\frac{dT}{dr}$$

(The effect is slightly less for higher mass stars, since the amount of energy generation is proportional to Z in CNO burning, but the smaller opacity is still the most important factor.) The luminosity increase also results in a hotter surface temperature, via $\mathcal{L} \propto R^2T_{\rm eff}^4$

• Changes in the metallicity and helium abundance also affect the precise mass where the convective core disappears. Greater helium abundance forces ρT to be larger to maintain the pressure. This produces an increase in luminosity, a larger value of $\nabla_{\rm rad}$, and an increased tendency towards convection. Similarly, a decrease in Z translates into a lower opacity, a larger luminosity, and a greater probability of convection.

• The central pressure is inversely proportional to stellar mass, as predicted by homology. Similarly, the central temperature also behaves as predicted, *i.e.,* it is directly proportional to mass. At large and small masses, the central density also obeys its homology relation, but the central density becomes almost independent of mass near $\sim 1\mathcal{M}_\odot$. This behavior is a natural consequence of the virial theorum, the main-sequence mass-luminosity relation, and the different temperature sensitivities of the CNO and pp chains.

Since main sequence stars are in virial equilibrium,

$$T_c \propto \frac{\mathcal{M}}{R} \implies T_c^3 \propto \left(\frac{\mathcal{M}}{R}\right)^3 \propto \rho_c\mathcal{M}^2 \tag{18.1}$$

Meanwhile, the nuclear energy generation rate is

$$\frac{d\mathcal{L}}{d\mathcal{M}} = \epsilon_n \propto \rho_c T_c^\nu \implies \mathcal{L} \propto \rho_c T_c^\nu \mathcal{M} \tag{18.2}$$

while the mass-luminosity relation is

$$\mathcal{L} \propto \mathcal{M}^{\eta} \tag{18.3}$$

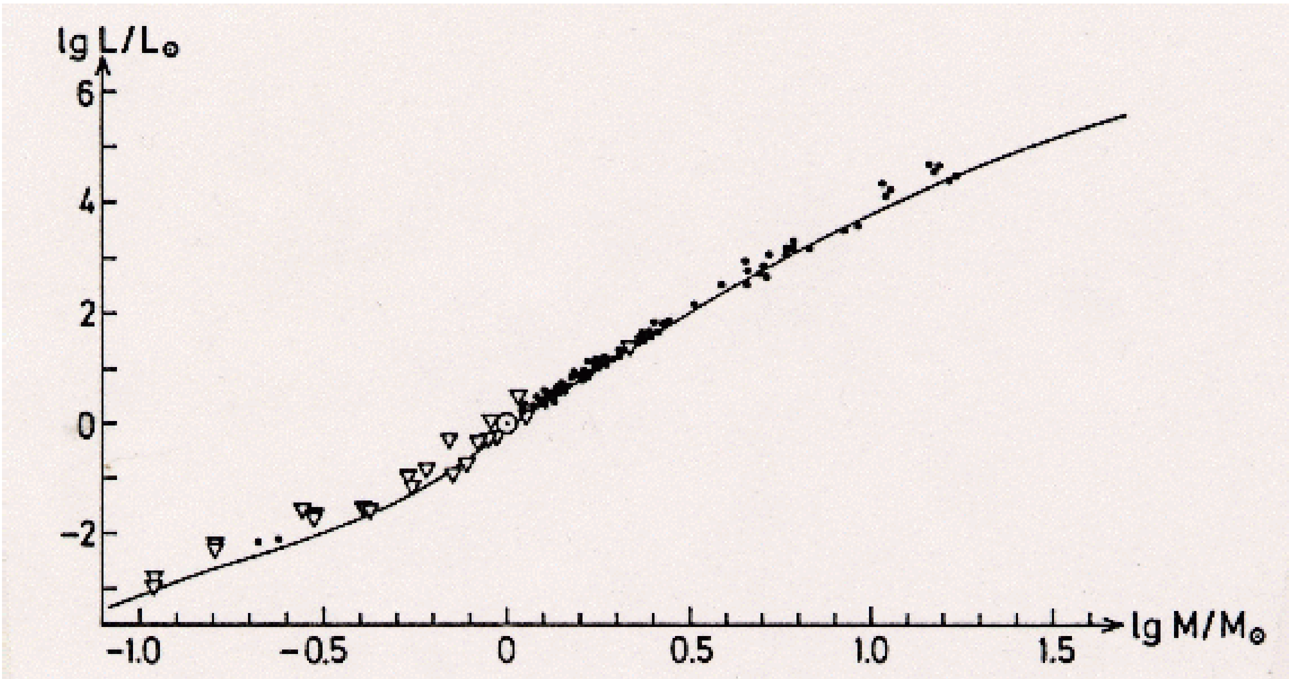
Now, if we substitute in for temperature using (18.1) and luminosity using (18.3), then

$$\mathcal{M}^{\eta} \propto \rho_c T_c^{\nu} \mathcal{M} \propto \rho_c \left(\rho_c^{1/3} \mathcal{M}^{2/3}\right)^{\nu} \mathcal{M}$$

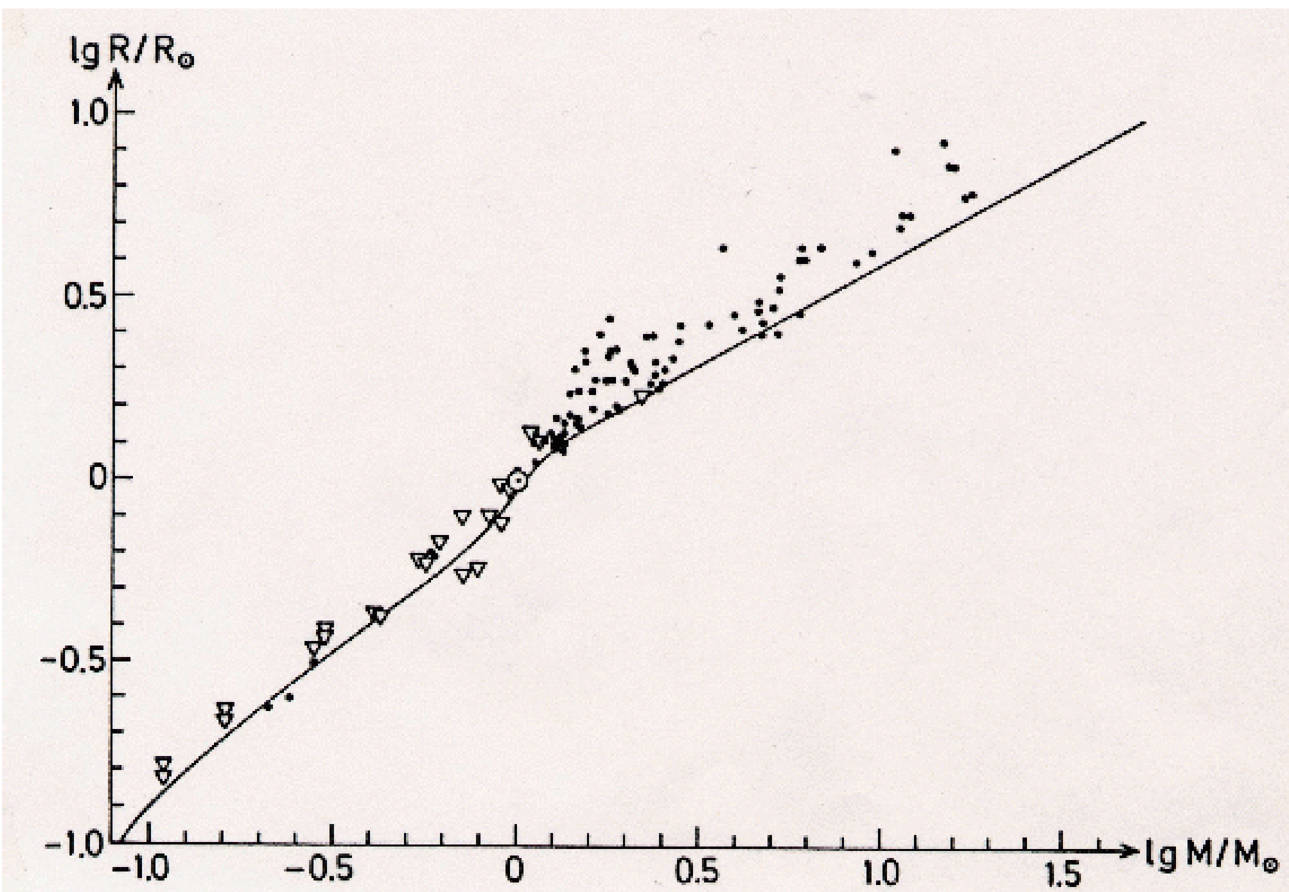
or

$$\rho_c \propto \mathcal{M}^{\frac{3\eta - 3 - 2\nu}{3+\nu}} \tag{18.4}$$

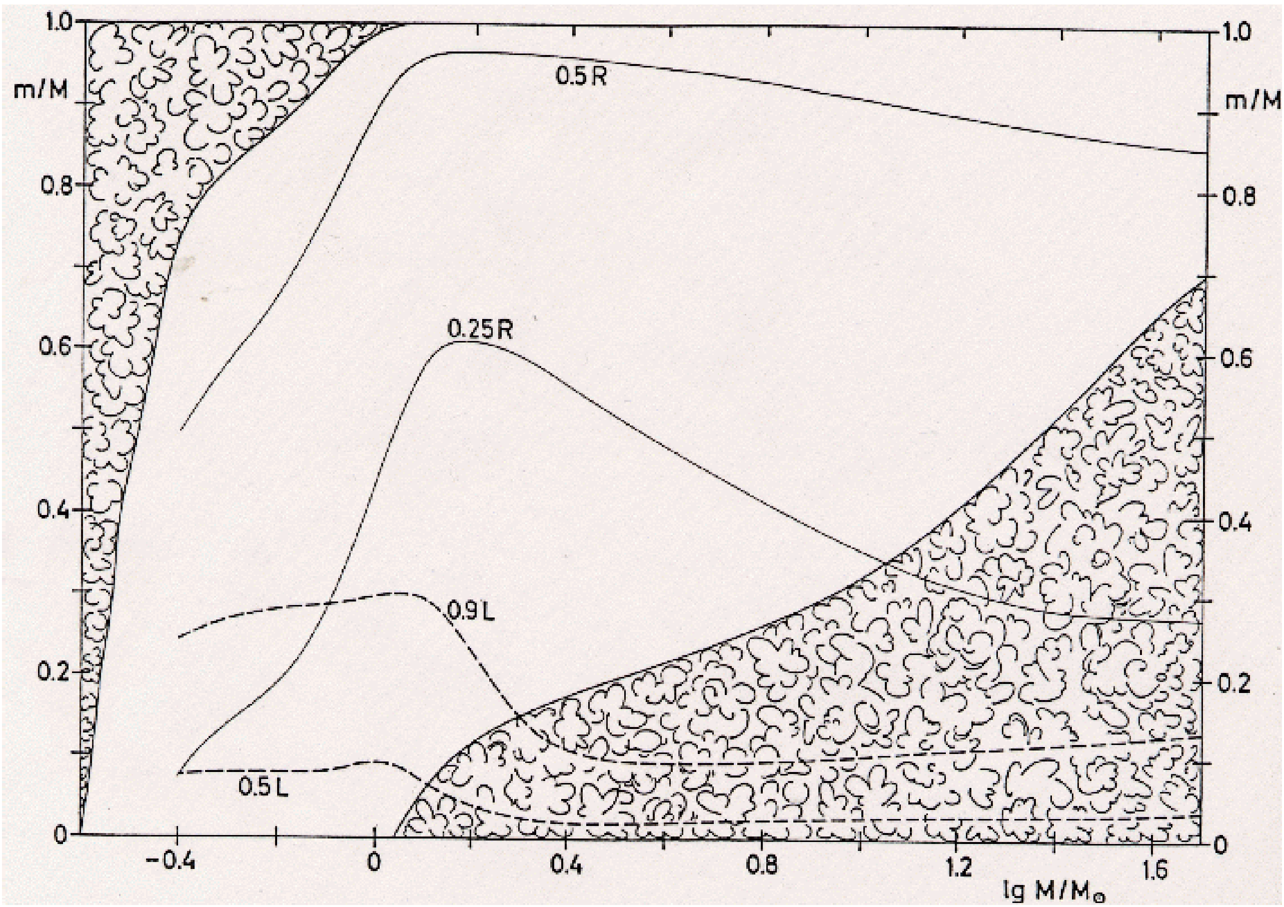
At high masses, $\nu \sim 18$ and $\eta \sim 3$, so the exponent is negative; at low masses; $\nu \sim 5$, and $\eta \sim 2$, and again, the exponent is negative. However, around $1\mathcal{M}_{\odot}$, $\nu \sim 4$, and $\eta \sim 4$; this generates a positive exponent, and thus changes the behavior of the central density.



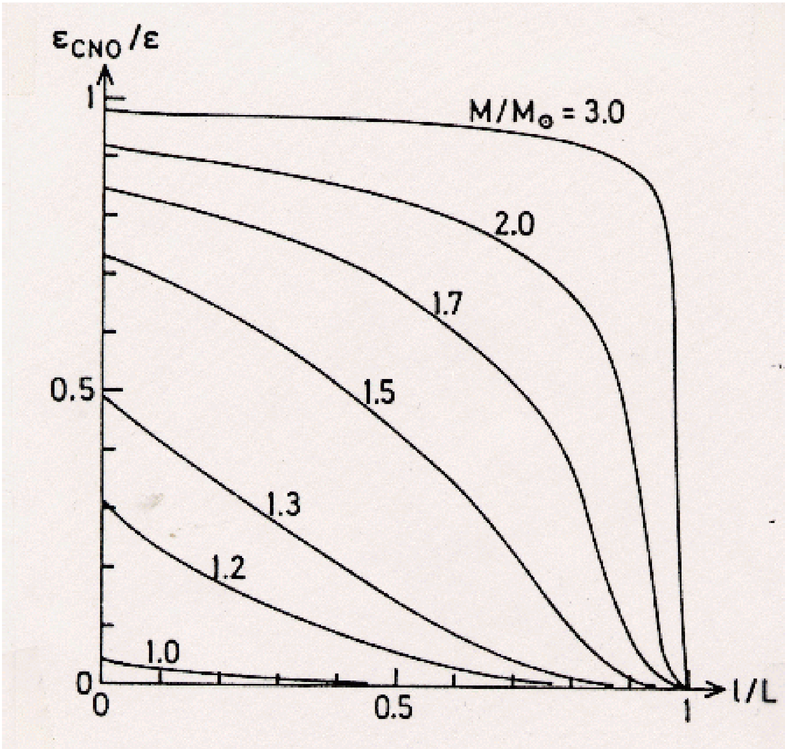
The mass-luminosity relation for main-sequence stars.



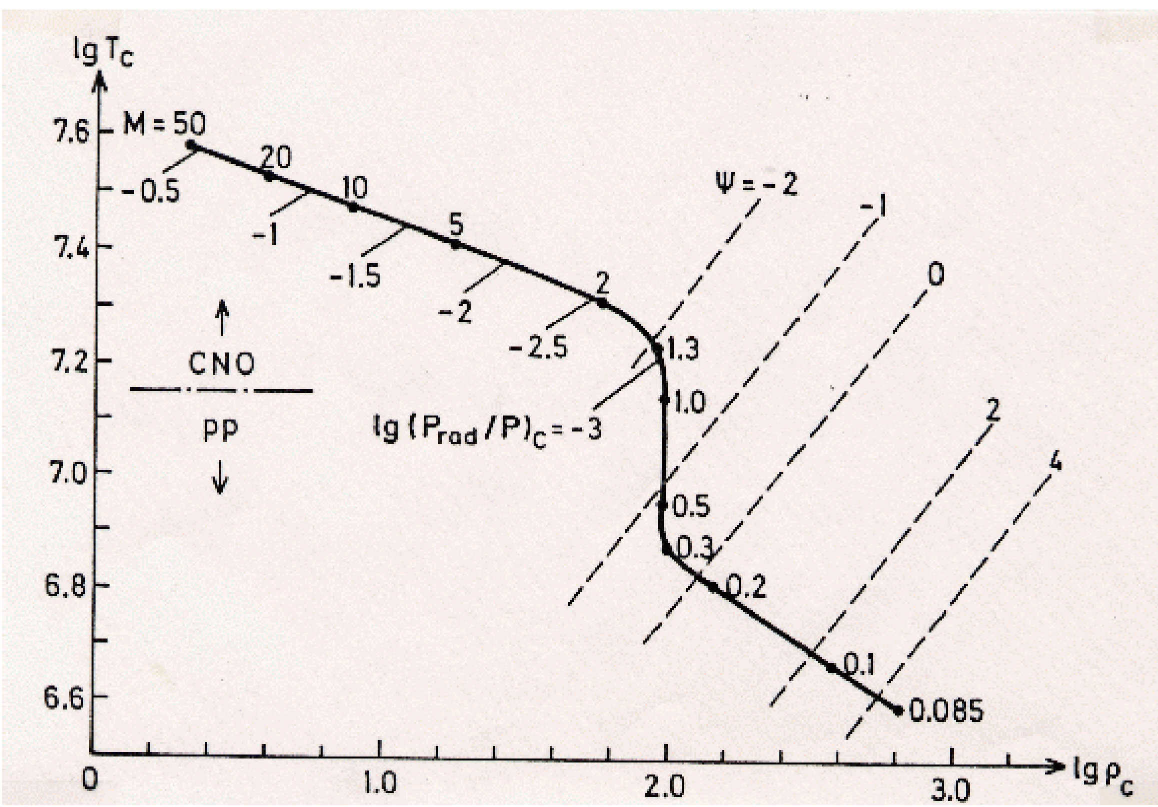
The mass-radius relation for main-sequence stars.



The mass fraction of main sequence stars plotted against the total stellar mass. “Cloudy” areas indicate the extension of convection zones. The solid lines give the mass values at 1/4 and 1/2 the stellar radius. The dash lines show the mass within which 50% and 90% of the stellar luminosity is produced.



The fraction of CNO energy generated for stars of different masses.



The central temperature and density for main sequence stars.

Main Sequence Luminosity Evolution

As a star burns on the main sequence, its luminosity will increase. This is due to the star's higher mean molecular weight: since the core has fewer particles to support it, ρT must increase, and so must the rate of nuclear reactions. To estimate the amount and timescale for this brightening, first consider that according to the virial theorum

$$nkT = \frac{G\mathcal{M}}{R} \Longrightarrow \frac{1}{\mu}T \propto \frac{\mathcal{M}}{R}$$

or

$$T \propto \mu \mathcal{M}^{2/3} \rho^{1/3} \tag{18.5}$$

where we have substituted for R using $\rho \propto \mathcal{M}/R^3$. Now if we assume that energy is transported via radiation, then from (3.1.4),

$$\mathcal{L} = -\frac{16acr^2T^3}{3\kappa\rho}\frac{dT}{dr}$$

Very roughly,

$$\frac{dT}{dr} \sim \frac{T - T_0}{0 - R} \approx -\frac{T}{R}$$

so

$$\mathcal{L} \propto \frac{RT^4}{\kappa\rho} \tag{18.6}$$

For main sequence stars, Kramers law opacity dominates (except perhaps in the very center, where electron scatter is important). Thus,

$$\mathcal{L} \propto \frac{RT^{15/2}}{\rho^2}$$

If we again substitute for R using $\rho \propto \mathcal{M}/R^3$, and T using (18.5), then

$$\mathcal{L} \propto \frac{\mathcal{M}^{1/3} T^{15/2}}{\rho^{7/3}} \propto \frac{\mathcal{M}^{1/3} \mu^{15/2} \mathcal{M}^5 \rho^{5/2}}{\rho^{7/3}} \propto \mathcal{M}^{16/3} \rho^{1/6} \mu^{15/2} \tag{18.7}$$

Since the mass of the star is (essentially) constant, and the density dependence is very weak, (18.7) says that

$$\frac{\mathcal{L}(t)}{\mathcal{L}_0} = \left[\frac{\mu(t)}{\mu_0} \right]^{\psi} \tag{18.8}$$

where $\psi = 15/2$. (Note that the homology equation for a Kramer law opacity gives gives $8.2 < \psi < 9.0$, depending on the precise value of ν.) Next, consider the definition of mean molecular weight. From (5.1.5) and (5.1.6), the mean molecular weight of the electrons is

$$\mu_e = \left(\sum_i \frac{Z_i x_i f_i}{A_i} \right)^{-1} \approx \frac{2}{1+X}$$

Meanwhile, the mean molecular weight of ions is

$$\mu_I = \left(\sum_i \frac{x_i}{A_i} \right)^{-1} \approx \left(X + \frac{Y}{4} + \frac{Z}{\sim 14} \right)^{-1} \approx \left(X + \frac{1-X}{4} \right)^{-1}$$

which reduces to

$$\mu_I \approx \frac{4}{1+3X} \tag{18.9}$$

The total mean molecular weight, as a function of hydrogen fraction, is therefore

$$\mu(t) \approx \left[\frac{1}{\mu_I} + \frac{1}{\mu_e} \right]^{-1} = \frac{4}{3+5X} \tag{18.10}$$

Finally, if the fusion process releases $Q = 6 \times 10^{18}$ ergs for every gram of hydrogen fused, then the rate of fuel consumed is related to the star's luminosity by

$$\frac{dX}{dt} = -\frac{\mathcal{L}}{\mathcal{M}Q} \tag{18.11}$$

Through equations (18.8), (18.10), and (18.11), we can compute the how luminosity of a main sequence star changes with time, *i.e.*,

$$\frac{d\mathcal{L}}{dt} = \left(\frac{d\mathcal{L}}{d\mu}\right)\left(\frac{d\mu}{dX}\right)\left(\frac{dX}{dt}\right) \tag{18.12}$$

where

$$\frac{d\mathcal{L}}{d\mu} = \psi\,\left(\frac{\mathcal{L}_0}{\mu_0}\right)\left(\frac{\mu}{\mu_0}\right)^{\psi-1} \tag{18.13}$$

and

$$\frac{d\mu}{dX} = -5\,\frac{4}{(3+5X)^2} = -\frac{5}{4}\mu^2 \tag{18.14}$$

Putting it all together, the expression for $\mathcal{L}$ becomes

$$\begin{aligned}
\frac{d\mathcal{L}}{dt} &= \left\{\frac{\psi\mathcal{L}_0}{\mu_0}\mu^{\psi-1}\right\}\left\{\frac{5}{4}\,\mu^2\right\}\left\{\frac{\mathcal{L}}{\mathcal{M}Q}\right\} \\
&= \frac{5}{4}\,\psi\,\mathcal{L}_0\;\mu_0\left(\frac{\mu}{\mu_0}\right)^{\psi+1}\frac{\mathcal{L}}{\mathcal{M}Q} \\
&= \frac{5}{4}\,\psi\;\mathcal{L}_0\;\mu_0\left(\frac{\mathcal{L}}{\mathcal{M}Q}\right)\;\left(\frac{\mathcal{L}}{\mathcal{L}_0}\right)^{(\psi+1)/\psi}
\end{aligned}$$

$$\frac{d\mathcal{L}}{dt} = \frac{5\,\psi\,\mu_0}{4\,\mathcal{L}_0^{1/\psi}\mathcal{M}Q}\,\mathcal{L}^{2+\frac{1}{\psi}} \tag{18.15}$$

This is easily integrated to give

$$\mathcal{L}^{-(\psi+1)/\psi} - \mathcal{L}_0^{-(\psi+1)/\psi} = -\frac{5\,(\psi+1)\,\mu_0}{4\mathcal{L}_0^{1/\psi}\mathcal{M}Q}t$$

or

$$\mathcal{L}(t) = \mathcal{L}_0\left\{1 - \frac{5}{4}\,(\psi+1)\,\frac{\mu_0\mathcal{L}_0}{\mathcal{M}Q}\;t\right\}^{-\psi/(\psi+1)} \tag{18.16}$$

If $\mu_0 \approx 0.6$ and $\psi = 15/2$, then in solar units, we have

$$\frac{\mathcal{L}}{\mathcal{L}_\odot} = \frac{\mathcal{L}(0)}{\mathcal{L}_\odot}\left\{1 - 0.29\,\frac{\mathcal{L}(0)}{\mathcal{L}_\odot}\,\frac{t}{t_\odot}\right\}^{-15/17} \tag{18.17}$$

For the sun, this relation gives $\mathcal{L}_\odot \approx 1.26\mathcal{L}_0$; if the homology exponents are used, the value changes to between 1.28 and 1.31 This compares to detailed models which imply that the sun has brightened by a factor of 1.37. (In other words, the sun as a ZAMS star was $\sim 25\%$ less luminous than it is today.)

Other Facets of Main Sequence Evolution

• During main sequence evolution, the increase in core temperature increases the pressure just outside the core. This (through some complicated physics), causes the surrounding regions to expand. (This in part explains why our estimate of the solar luminosity increase overestimates the actual number. Some of the energy has gone into expansion.) *In general, any time a stellar core contracts, the envelope surrounding the core will expand.*

• In high mass stars, the CNO nuclear reactions are highly concentrated in the center of the star. As a result, the molecular weight is unchanged over most of the star, and only the core contracts. The star therefore becomes redder, as the increase in the radius overwhelms that of the luminosity. For low mass stars, the relatively weak dependence of pp-cycle energy generation on temperature allows more of the star to be involved in nuclear reactions. A larger fraction of the star thus has its molecular weight changed, and more of the star participates in the contraction. This competes with the overall stellar expansion. (A $1\mathcal{M}_\odot$ main-sequence star actually becomes hotter with age.)

• As a fully convective star ($\mathcal{M} < 0.3\mathcal{M}_\odot$) evolves, the change in its molecular weight just moves the star to a higher helium content ZAMS model.

• Convective cores in stars with $\mathcal{M} < 2.25\mathcal{M}_\odot$ are small, but they increase (slightly) with time. As the core temperature increases, the CN-cycle becomes more important, driving more convection. Eventually, however, the convective core will again become smaller, as the decrease in Kramer's opacity takes over.

Semi-Convection

For stars with $\mathcal{M} > 2.25\mathcal{M}_\odot$, the mass fraction of the convective core decreases with time. This is due to the decrease in the opacity with temperature, which follows from Kramer's law. As the convective core gets smaller, it leaves behind a chemical gradient. This causes a phenomenon known as semi-convection.

Consider a star with a convective core within which nuclear reactions are proceeding, and a radiative envelope. In the convective core, $\nabla_{\rm rad} > \nabla_{\rm ad}$; outside the core, $\nabla_{\rm rad} < \nabla_{\rm ad}$, and, at the border between the two regions, $\nabla_{\rm rad} = \nabla_{\rm ad}$. As always, $\nabla_{\rm rad}$ is given by

$$\nabla_{\rm rad} = \frac{3\kappa\mathcal{L}P}{16\pi acG\mathcal{M}T^4} \tag{3.1.6}$$

Now assume that the major source of opacity is electron scattering, and that the matter is fully ionized. (Since we are talking about the core of a massive star, these are excellent assumptions.) As nuclear reactions proceed in the core, the hydrogen content of the region will decrease. Since the electron scattering opacity for fully ionized matter is given by

$$\kappa_e = 0.2(1 + X) \tag{6.1.7}$$

this results in a decrease in the stellar opacity.

Now let's examine the behavior of $\nabla_{\rm rad}$ at a distance ϵ from the convective core. By definition, for convection to occur

$$\nabla_{\rm rad,i} > \nabla_{\rm ad} \quad \text{inside the core}$$

and

$$\nabla_{\rm rad,o} < \nabla_{\rm ad} \quad \text{outside the core}$$

Therefore, since $\nabla_{\rm ad}$ shouldn't change much over a distance ϵ, $\nabla_{\rm rad,i} > \nabla_{\rm rad,o}$ by the definition of convection. However, $\nabla_{\rm rad}$ is directly proportional to the hydrogen content, and $X_i < X_{\rm o}$. Thus, $\nabla_{\rm rad,i} < \nabla_{\rm rad,o}$ from the opacity. In other words, there is a fundamental contradiction.

Another way to look at the problem is to consider the temperature and density difference between a convective blob and its surroundings. Recall that for a region to be dynamically stable,

$$\left(\frac{d\rho}{dr}\right)_i > \left(\frac{d\rho}{dr}\right)_s \tag{3.2.1}$$

or

$$\frac{\Delta\rho}{\Delta r} > 0$$

where Δ represents the difference between the region internal to the blob and the blob's surroundings. For positive displacements of Δr, the blob will be denser than its surroundings, and will sink back to its original position.

Now consider the temperature difference between the blob and the surrounding.

$$\Delta T = \left[\left(\frac{dT}{dr}\right)_i - \left(\frac{dT}{dr}\right)_s\right]\Delta r$$

If the blob is in pressure equilibrium (given) and the stellar region is chemically homogeneous, then from (3.2.2)

$$\frac{\Delta T}{T} = -\frac{\Delta\rho}{\delta\rho}$$

The negative sign indicates that the inside of the blob will be cooler than its surroundings. It will therefore gain some energy

by radiation, which will result in the blob having a slightly lower density, and a slightly weaker restoring force due to buoyancy. Any motion of the blob will eventually be damped out.

Next, consider the situation where the surrounding medium is chemically inhomogeneous. We again assume that the region is dynamically stable, with $\Delta\rho/\Delta r > 0$. However, in this case

$$\frac{\Delta T}{T} = -\frac{\Delta\rho}{\delta\rho} + \frac{\varphi}{\delta}\frac{\Delta\mu}{\mu} \tag{8.2.1}$$

Consequently, if

$$\Delta\mu = \left(\frac{d\mu}{dr}\right)_i - \left(\frac{d\mu}{dr}\right)_s > \frac{\Delta\rho}{\rho}\frac{\mu}{\varphi}$$

then blobs that have a positive Δr displacement will have $\Delta T > 0$. These blobs will be hotter than their surroundings and will therefore lose heat as they move. When this happens, their density will increase, the restoring force will increase, and their oscillation will slowly increase in amplitude. The blob will be vibrationally unstable; the result will be a slow mixing, which will eventually destroy the chemical gradient. This semi-convection will occur when

$$\nabla_{\rm ad} < \nabla < \nabla_{\rm ad} + \frac{\varphi}{\delta}\left(\frac{d\ln\mu}{d\ln P}\right) \tag{8.2.2}$$

Note that from (8.2.1) and (8.2.2), it will only occur when the mean molecular weight is decreasing rapidly with radius, or equivalently,

$$\left(\frac{d\ln\mu}{d\ln P}\right) > 0$$

This occurs in high mass stars with convective cores. In these objects, the convective region gets smaller with time, leaving behind a chemical gradient.